- Office hours Mon 11-12
- HW posted; submit in groups of 3-4.
- Script volunteers → count as 1 HW.

Clarifications (Thanks to all who pointed out)

(1) $\boxed{\text{Abbildung}}$ = Map $\boxed{\text{Karte}}$ = Chart

Key ideas for today

* Tangent space already tells you alot abt mfld
  derivative — — — — map

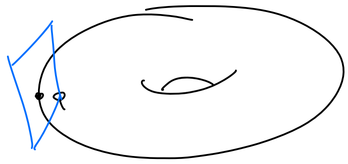
Next 2 lectures

* ways of building manifolds from other manifolds

For today, mfld = submfld of $\mathbb{R}^n$

$$\begin{array}{ccc} S \subseteq \mathbb{R}^n \text{ is locally} & \xRightarrow{\text{Monday}} & S \subseteq \mathbb{R}^n \text{ is a} \\ \text{a graph } \mathbb{R}^d \to \mathbb{R}^{n-d} & \xLeftarrow[\text{Today}]{} & d\text{-manifold} \end{array}$$

Pf



We need a way to choose the d coords

all we have are $\varphi : \underset{\mathbb{R}^d}{\overset{\cap}{U}} \to S$ and $\varphi^{-1} : \underset{\mathbb{R}^n}{\overset{\subseteq}{V}} \to U$

Tangent space

Prop $d\varphi_q : \mathbb{R}^d \to \mathbb{R}^n$ is injective $\forall q$

Pf $d\varphi^{-1} \circ d\varphi : \mathbb{R}^d \to \mathbb{R}^d = \mathbb{1}$ ⌟ Chain rule!

Prop Image $(d\varphi_{\varphi^{-1}(p)})$ does not depend on the chart

Pf

$$\begin{tikzcd} & S & \\ U \arrow[ur, "\varphi"] & & \tilde{U} \arrow[ul, "\tilde{\varphi}"'] \arrow[ll, "\varphi^{-1}\circ\tilde{\varphi}"] \end{tikzcd}$$

all invertible

$d(\tilde{\varphi}) = d\varphi \circ d(\varphi^{-1} \circ \tilde{\varphi})$ ⌟

(note φ^{-1} does not determine T_pS)

Dfn $T_pS :=$ Image of $d\varphi_{\varphi^{-1}(p)}$

Rmk $d\varphi : \mathbb{R}^d \to T_pS$ is an isomorphism

Cor If S is locally a graph over $\mathbb{R}^d$, then $\pi : \mathbb{R}^n \to \mathbb{R}^d$

$d\pi : T_pS \xrightarrow{d} \mathbb{R}^d$ is an isomorphism.

Pf π is the inverse of a chart ⌟

For the converse, need:

<u>Thm</u> (Inverse Function theorem) $f : \underset{\text{op}}{\mathcal{U}^{\subseteq \mathbb{R}^d}} \longrightarrow \mathbb{R}^d$ smooth

If $df_p : \mathbb{R}^d \to \mathbb{R}^d$ is invertible, then f is locally (smoothly) invertible at p. (a local diffeomorphism)

Warning: local maps $\mathcal{U} \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \mathbb{R}^d$ may be vague abt domain; if you like, everything today is about germs of maps.

End of pf of graph thm:

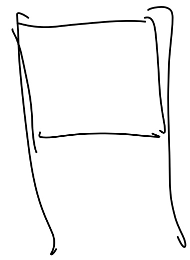
Let $p \in S$, $\psi : \mathcal{U}^{\subseteq \mathbb{R}^d} \to S$ be a chart at p;

$d\psi$ injective

$\Rightarrow$ some $d \times d$ minor is invertible

Choose this π_d

$$\begin{array}{cc} d & \\ \begin{bmatrix} \\ d\psi \\ \\ \end{bmatrix} & n \end{array}$$

WLOG 

then $d(\pi_d \circ \psi)$ is an isomorphism

$\xrightarrow{\text{IFT}}$ It has a local inverse f

- $(\pi_d \circ \psi) \circ f = 1_V$
- $f \circ (\pi_d \circ \psi) = 1_{\mathcal{U}}$

$$\begin{tikzcd} & S \arrow[dl, "\pi_d"'] & \\ V \arrow[rr, "f"'] & & \mathcal{U} \arrow[ul, "\psi"'] \end{tikzcd}$$

ψ^{-1}

<u>Claim</u> $(\psi \circ f \quad)$ presents S locally as a graph.

- $\pi_d \circ (\psi \circ f) = id_V \Rightarrow \psi \circ f(x_1 \ldots x_d) = (x_1 \ldots x_d, (\psi \circ f)_{d+1}(x_1 \ldots x_d) \ldots (\psi \circ f)_n(x_1 \ldots x_d))$

need
- $\psi \circ f \circ \pi_d \big|_S = id_S$ locally

← conjugate by ψ

- dom/codom
- word definition

Ways of getting new manifolds from old

- products (HW)

$S \subseteq \mathbb{R}^n$, $\hat{S} \subseteq \mathbb{R}^m$ $\longrightarrow$ $S \times \hat{S} \subseteq \mathbb{R}^{n+m}$

- graphs $\longrightarrow$ level sets (Gauss)

$\downarrow$ generalizes to

parametrizations

Images/ embeddings / parametrizations

<u>Defn</u>: $\varphi : \mathcal{U}^{\subseteq \mathbb{R}^d} \longrightarrow \mathbb{R}^n$ is an embedding if $\varphi : \mathcal{U} \longrightarrow \varphi(\mathcal{U})$ is a diffeomorphism.

<u>Defn</u>: immersion

(so emb $\Rightarrow$ imm, but not conversely

That a <u>graph</u> is an embedding, so:

<u>Thm</u> If $\varphi : \mathcal{U}^{\subseteq \mathbb{R}^d} \longrightarrow \mathbb{R}^n$ is an immersion, then it is locally (in $\mathcal{U}$) an embedding.

(note similarity & differences to above)

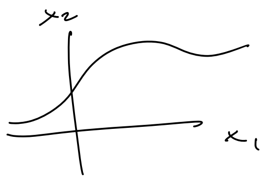
$\ulcorner$ As above, choose $\mathbb{R}^d \subseteq \mathbb{R}^n$ s.t.

$$\begin{array}{ccc} & & \mathbb{R}^n \\ & \overset{\varphi}{\nearrow} & \downarrow \text{proj} \\ \mathcal{U} & \xleftarrow{f} & \mathbb{R}^d \end{array}$$

locally invertible

then $\varphi \circ f \big|_{\text{dom } f}$ is a graph (on the whole domain), so it's an embedding.

Straightening a graph example:



$x_2 = f(x_1)$

$\tilde{x}_1 = x_1$

$\tilde{x}_2 = x_1 - f(x_1)$

$\Rightarrow \left[\frac{\partial \tilde{x}_i}{\partial x_j}\right] = \begin{bmatrix} 1 & 0 \\ -f' & 1 \end{bmatrix}$ invertible (eg determinant check)

HW: (canonical form of an immersion)

every immersion, in some coords, is the inclusion $\iota_d : \mathbb{R}^d \to \mathbb{R}^n$.

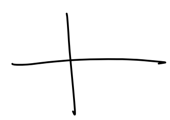
Submersion / Level sets

(heuristic — condition decreases dim by 1)

$U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}^{n-d}$

when is $f^{-1}(0)$ a d-submfld of U?

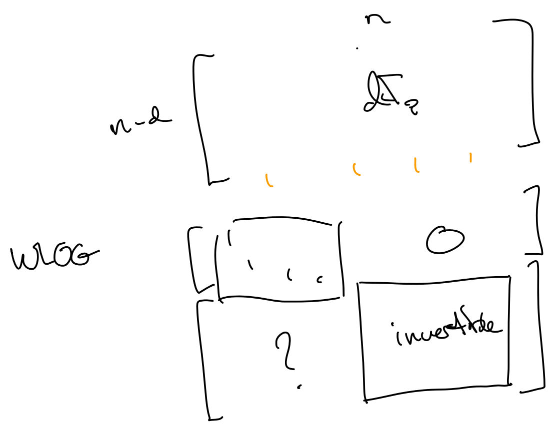
non-ex $f(x,y) = xy$



$f(x,y) = x^2+y^2$

Thm For $p \in f^{-1}(0)$, if df_p is surjective, then $f^{-1}(0)$ is a manifold near p.

Pf



get map $\mathbb{R}^n \longrightarrow \mathbb{R}^d \times \mathbb{R}^{n-d}$

$x_1 \ldots x_n \mapsto (x_1 \ldots x_d) \times (f_{d+1} \ldots f_n)$

IFT $\longrightarrow$ local inverse $G(x_1 \ldots x_d, f_1 \ldots f_n)$

then $G(x_1 \ldots x_d, 0 \ldots 0)$ is a chart

w/ inverse $(x_1, \ldots, x_d)$

Another way of saying this; IFT $\Rightarrow$ $(x_1 \ldots x_d, f_{d+1} \ldots f_n)$ are local coords. They clearly straighten fibers.

<u>Thm</u> (canonical form of a submersion) $\exists$ coords s.t. the submersion is π_{red}:

<u>Note</u> If f is a submersion, $T_p f^{-1}(0) = \ker(df_p)$

<u>Thm</u>: If $f : U^{\subseteq \mathbb{R}^n} \longrightarrow \mathbb{R}^m$, $S^{m-r} \subseteq \mathbb{R}^m$ submfld, and $\text{Im}(df_p) \oplus T_{f(p)} S = \mathbb{R}^m$, then $f^{-1}(S)$ is a $(n-r)$-submfld.

$\ulcorner$
- WLOG, near p, $S = \mathbb{R}^{m-r} \subseteq \mathbb{R}^m$ (fund thm of imm)
- $S \xrightarrow{\pi_r} \mathbb{R}^r$ is a submersion at p $\lrcorner$

<u>Dfn</u> (Transversality)
- If $f : U \to \mathbb{R}^m$, $S \subseteq \mathbb{R}^m$
  $f \pitchfork S$ if $\text{Im}(df_p) \oplus T_{f(p)} S = \mathbb{R}^m$ $\forall p \in f^{-1}(S)$
- If $S, \hat{S} \subseteq \mathbb{R}^m$, $S \pitchfork \hat{S}$ if $T_p \hat{S} \oplus T_p S = \mathbb{R}^m$ $\forall p \in S \cap \hat{S}$
- If $f, g : U \to \mathbb{R}^m$, $f \pitchfork g$ if $\forall p \in U$ s.t. $f(p) = g(p)$,
  $df_p - dg_p : T_p U \twoheadrightarrow \mathbb{R}^m$

Implication

- $f \pitchfork S \Rightarrow f^{-1}(S)$ submfld
- $S \pitchfork \hat{S} \Rightarrow S \cap \hat{S}$ submfld (apply to $f = \iota : \hat{S} \hookrightarrow \mathbb{R}^n$)
- $f \pitchfork g \Rightarrow \{f = g\}$ submfld (dim $n-m$)

$$\begin{tikzcd} U^{\subseteq \mathbb{R}^n} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbb{R}^m \end{tikzcd}$$

⌜ $(f,g) \pitchfork \Delta$ where

$(f,g) : U \to \mathbb{R}^{2m}$, $\Delta \subseteq \mathbb{R}^{2m}$ is the diagonal ⌟

Rmk Everything makes sense with $\mathbb{R}^n$ replaced by an n-dim'l mfld, e.g.

Thm (IFT for submanifolds)

If $S, \hat{S}$ submflds, $df_p : T_pS \longrightarrow T_{f(p)}\hat{S}$ is an isomorphism, then f is a local diffeomorphism.

⌜
$$\begin{tikzcd} S \arrow[r, "f"] & \hat{S} \\ U \arrow[u, "\psi"] \arrow[r, shift left] & \hat{U} \arrow[u, "\hat{\psi}"'] \arrow[l, shift left, dashed, "\text{IFT}"] \end{tikzcd}$$
⌟

and

if $S \underset{g}{\overset{f}{\rightrightarrows}} X$, say f transverse to g ($f \pitchfork g$)

if $\forall p \in S$ w/ $\underbrace{f(p) = g(p)}_{=: q}$, $\mathrm{Image}(df_p - dg_p) = T_q X$

Thm If $f \pitchfork g$, then $\{f = g\}$ is a submfld of S

[• if $X = \mathbb{R}^n$, can apply submersion thm to $f - g$.

• In general, choose a local diffeo $X \simeq \mathbb{R}^n$]

Note If $S, \hat{S} \subseteq X$, then $S \pitchfork \hat{S} \iff \iota_S \pitchfork \iota_{\hat{S}}$